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The Thermodynamics of the Curie–Weiss Model with Random Couplings

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We study the Curie–Weiss version of an Ising spin system with random, positively biased couplings. In particular, the case where the couplings ε_{ij} take the values one with probability p and zero with probability $1-p$, which describes the Ising model on a random graph, is considered. We prove that if p is allowed to decrease with the system size N in such a way that $Np(N)\uparrow\infty$ as $N\uparrow\infty$, then the free energy converges (after trivial rescaling) to that of the standard Curie–Weiss model, almost surely. Similarly, the induced measures on the mean magnetizations converge to those of the Curie–Weiss model. Generalizations of this result to a wide class of distributions are detailed.

KEY WORDS: Curie–Weiss model; random graphs; disordered magnets; mean-field theory.

1. INTRODUCTION

In recent years there has been a revival of interest in the Curie–Weiss (CW) model (also called "mean-field model") of ferromagnets and some of its derivatives (e.g., the Curie–Weiss–Potts model, etc.).[6,7] The use of large-deviation techniques (see, e.g., the book by Ellis[5] for a review in this context) has made it possible to give a very neat probabilistic description of the thermodynamic limit for such models, which has made them some of the best understood models in statistical mechanics. Interestingly, these techniques have even allowed one to treat several types of *disordered* CW models: Amaro de Matos and Perez[14] have analyzed the CW model with a random magnetic field term and not only constructed the thermodynamic limit, but also obtained results on the fluctuations of various thermo-

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dynamics quantities. The infinite-volume Gibbs states for this model have been studied recently by Amaro de Matos *et al.*[15] Another model that has been solved exactly is the Hopfield model of neural networks (under some restriction on the number of stored patterns), which can be seen as a CW model with a particular type of random exchange coupling.[11,9] Of course, the possibly most celebrated mean-field model, the Sherrington–Kirkpatrick model[19] for spin-glasses, still awaits a rigorous mathematical analysis, in spite of many efforts and the existence even of an "exact" solution based on what is called the "replica symmetry breaking scheme" (for a review and references see, e.g., the book by Mézard *et al.*[16]).

A simpler model than spin glasses, but nonetheless one with genuine "bond disorder," is the so-called "dilute ferromagnet" (see, e.g., Fröhlich's lecture[8] for a review). Here the exchange couplings between spins are random, but strictly (or at least predominantly) ferromagnetic. Using techniques from percolation theory, it has been proven[10,1] that at low temperatures this model exhibits a ferromagnetic phase, provided only the nonzero bonds percolate. On the other hand, critical properties of this model, in particular in dimension $d=2$, are heavily disputed in the physics literature.[4,18,13,23] Surprisingly enough, it appears that the CW version of this model has so far not been investigated, and this is what we propose to do in this article. More precisely, we will show that under some (fairly weak and natural) assumptions on the disorder, an exact solution in terms of the quantities of the standard CW model can be given. It should be noted that our present results are, in probabilistic language, on the level of "laws of large numbers"; fluctuation theorems will be left to further investigation.

To be able to state our results in a precise way let us given a definition of the models we will treat. For a given positive integer N, let Λ denote the set $\Lambda=\{1,..., N\}$. To each site $i\in\Lambda$ is associated an Ising spin variable $\sigma_i\in\{-1, 1\}$ and a spin configuration on Λ is given by the sequence $\sigma=\{\sigma_i\}_{i\in\Lambda}$. The configuration space is denoted by $\Gamma^N=\{-1, 1\}^N$. We recall that in the standard CW model the interaction energy of a spin configuration $\sigma\in\Gamma^N$ is obtained by coupling each pair of spins at sites (i, j) in $\Lambda\times\Lambda$ with equal strength $1/(2N)$, that is to say

$$\begin{aligned} H_N^h(\sigma) &= -\sum_{(i,j)\in\Lambda\times\Lambda} J_{i,j}\sigma_i\sigma_j - h\sum_{i\in\Lambda}\sigma_i \\ &= -\frac{1}{2N}\sum_{(i,j)\in\Lambda\times\Lambda}\sigma_i\sigma_j - h\sum_{i\in\Lambda}\sigma_i \end{aligned} \tag{1.1}$$

where the external magnetic field h is a real number. The energy function of the randomly dilute CW model (RDCW) is obtained from (1.1) by choosing the couplings $J_{i,j}$ as random variables normalized such that $\mathbb{E}(J_{i,j}) = 1/(2N)$. An elementary implementation of the notion of dilution consists then in defining $J_{i,j} \equiv \varepsilon_{i,j}/2Np$, where $\varepsilon_N = \{\varepsilon_{i,j}\}_{i=1,...,N; j=1,...,N}$ are independent and identically distributed random variables (i.i.d.r.v.'s) with $\mathbb{P}(\varepsilon_{i,j} = 1) = 1 - \mathbb{P}(\varepsilon_{i,j} = 0) = p$. The Hamiltonian then reads

$$H_N^h(\varepsilon_N, \sigma) = -\frac{1}{2Np} \sum_{(i,j) \in \Lambda \times \Lambda} \varepsilon_{i,j} \sigma_i \sigma_j - h \sum_{i \in \Lambda} \sigma_i \tag{1.2}$$

While this setup suffices to define the RDCW model for a fixed N, since we are interested in taking limits as N goes to infinity later on, we need to be more specific on the random variables $\varepsilon_{i,j}$ as functions of N. This is somewhat more subtle than usual, due to the fact that we will allow p to be a function of N. There are several ways to set up the probabilistic environment for this. We prefer, however, the following: Let us fix a function $p: \mathbb{N} \to (0, 1]$ and let us first consider a fixed $(i, j) \in \mathbb{N} \times \mathbb{N}$. We introduce a probability space $(\Omega_{i,j}, \Sigma_{i,j}, \mathbb{P})$, with $\Omega_{i,j} \equiv \{0, 1\}^{\mathbb{N}}$, such that $\{\varepsilon_{i,j}(N)\}_{N \in \mathbb{N}}$ is an (inhomogeneous) Markov chain on this probability space, with transition probabilities given by

$$\begin{aligned}
&\mathbb{P}(\varepsilon_{i,j}(N) = 0 \mid \varepsilon_{i,j}(N-1) = 0) = 1 \\
&\mathbb{P}(\varepsilon_{i,j}(N) = 1 \mid \varepsilon_{i,j}(N-1) = 0) = 0 \\
&\mathbb{P}(\varepsilon_{i,j}(N) = 0 \mid \varepsilon_{i,j}(N-1) = 1) = 1 - q(N) \\
&\mathbb{P}(\varepsilon_{i,j}(N) = 1 \mid \varepsilon_{i,j}(N-1) = 1) = q(N)
\end{aligned} \tag{1.3}$$

where $q(N)$ is chosen such that $\mathbb{P}(\varepsilon_{i,j}(N) = 1) = p(N)$, that is, $q(N)$, that is, $q(N) \equiv p(N)/p(N-1)$. Note that this setup constrains p to be a non-increasing function of N. Now we introduce the product probability space $(\Omega, \Sigma, \mathbb{P}_\varepsilon)$:

$$(\Omega, \Sigma, \mathbb{P}_\varepsilon) \equiv \left(\prod_{i,j \in \mathbb{N} \times \mathbb{N}} \Omega_{i,j}, \prod_{i,j \in \mathbb{N} \times \mathbb{N}} \Sigma_{i,j}, \prod_{i,j \in \mathbb{N} \times \mathbb{N}} \mathbb{P} \right) \tag{1.4}$$

and we consider $\varepsilon_N \equiv \{\varepsilon_{i,j}(N)\}_{i=1,...,N; j=1,...,N}$ as a family of random variables on the product cylinder set $\{\omega \in \Omega: \omega_{i,j} \in \Omega_{i,j} \forall (i, j) \in \Lambda \times \Lambda\}$. From now on we write ε instead of ε_N provided there is no danger of confusion. The above construction has the virtue that it yields the maximal probability for $\varepsilon_{i,j}(N)$ to equal $\varepsilon_{i,j}(N-1)$ for the given marginals. In particular, this probability is one if $p(N)$ is constant.

The RDCW model is then defined by the probability measure $\mathcal{G}_N^{\beta,h}(\varepsilon)$ which assigns to each configuration $\sigma \in \Gamma^N$ the probability

$$\mathcal{G}_N^{\beta,h}(\varepsilon, \sigma) = \frac{\exp[-\beta H_N^h(\varepsilon, \sigma)]}{2^N Z_N^{\beta,h}(\varepsilon)} \tag{1.5}$$

where $\beta > 0$ is the inverse temperature and where the partition function $Z_N^{\beta,h}(\varepsilon)$ is given by

$$Z_N^{\beta,h}(\varepsilon) = \sum_{\sigma \in \Gamma^N} \frac{\exp[-\beta H_N^h(\varepsilon, \beta\sigma)]}{2^N} \tag{1.6}$$

Note that these last three quantities are random variables on the probability space $(\Omega, \Sigma, \mathbb{P}_\varepsilon)$. Before giving the statement of our main theorem, we need the following notations: let m_N be the block spin variable $m_N \equiv m_N(\sigma) \equiv (1/N) \sum_{i=1}^N \sigma_i$, $\mathscr{S}_N$ the set of all its possible values, $\mathscr{S}_N = \{-1, -1+2/N,..., 1-2/N, 1\}$, and $m_N^{\beta,h}(\varepsilon)$ the expectation of M_N with respect to $\mathscr{G}_N^{\beta,h}(\varepsilon)$. We will also exploit throughout the paper some well-known results of the standard CW model. For the sake of convenience we have summarized them in an appendix. The quantities referring to the standard CW model will be marked with a tilde. Finally, defining the finite-volume free energy as

$$f_N^{\beta,h}(\varepsilon) \equiv -\frac{1}{\beta N} \ln Z_N^{\beta,h}(\varepsilon) \tag{1.7}$$

we are ready to announce the following result.

Theorem 1. Let $p \in (0, 1]$ be a nonincreasing function of N such that $pN \uparrow \infty$ as $N \uparrow \infty$. Then, almost surely with respect to $\mathbb{P}_\varepsilon$, the following results hold:

(i) For all $\beta > 0$ and all $h \in \mathbb{R}$

$$\lim_{N \uparrow \infty} f_N^{\beta,h}(\varepsilon) = \tilde{f}_\infty^{\beta,h} \tag{1.8}$$

where $\tilde{f}_\infty^{\beta,h}$ is the infinite-volume free energy of the standard CW model.

Let $\mathscr{L}\{m_N\}$ be the law of m_N under $\mathscr{G}_N^{\beta,h}(\varepsilon)$ and let δ_x denote the Dirac measure concentrated on the point x. Denoting respectively by $\tilde{m}^{\beta,(+)}$ and $\tilde{m}^{\beta,(-)}$ the largest and smallest solutions of the equation $m = \tanh(\beta m)$, we have the following result.

(ii) For $h \geqslant 0$

$$\lim_{h \downarrow 0} \lim_{N \uparrow \infty} \mathscr{L}\{m_N\} = \begin{cases} \delta_0 & \text{if } 0 \leqslant \beta \leqslant 1 \\ \delta_{\tilde{m}^{\beta,(+)}} & \text{if } \beta \geqslant 1 \end{cases} \tag{1.9}$$

The same result holds for $h \leqslant 0$ with $\tilde{m}^{\beta,(+)}$ replaced by $\tilde{m}^{\beta,(-)}$.

(iii) For $h = 0$ and for all $\beta > 0$

$$\lim_{N \uparrow \infty} \mathscr{L}\{m_N\} = \tfrac{1}{2}\delta_{\tilde{m}^{\beta,(+)}} + \tfrac{1}{2}\delta_{\tilde{m}^{\beta,(-)}} \tag{1.10}$$

(but note that for $\beta \leqslant 1$, $\tilde{m}^{\beta,(-)} = \tilde{m}^{\beta,(+)} = 0$).

Remark. The condition $pN \uparrow \infty$ appears to be the weakest possible for the theorem to hold. It implies that the mean coordination number of each site goes to infinity as the system size diverges, that is, from a physical point of view the dimensionality of the system goes to infinity. On regular lattices, it has ben proven before (see, e.g., refs. 20, 21, 12, and 2) that the mean Curie–Weiss free energy is obtained in the limit of infinite dimension.

Remark. In Section 4 we will show that the analog of Theorem 1 can be proven for a much larger class of distributions of the ε_{ij}. For transparency and clarity we prefer, however, to first present the proof in this specific context.

The remainder of this paper is organized as follow: we show in Section 2 that the Hamiltonian (1.2) of the RDCW model can be seen, on a subset of Ω of $\mathbb{P}_\varepsilon$-measure one, as a small perturbation of the Hamiltonian (1.1) of the standard CW model. Therefore the proof of Theorem 1, given in Section 3, essentially follows from a standard mean-field treatment. An interesting issue of the method developed for the study of the RDCW model as defined in (1.2), (1.5) is that it applies for more general definitions of the random couplings $J_{i,j}$ and in particular does not require them to be ferromagnetic. In Section 4 we show how the method provides general conditions on the distribution of the coupling under which the results of Theorem 1 hold, and detail some specific examples, including Gaussian couplings.

2. BOUNDS ON THE HAMILTONIAN

The main idea behind the proof of the theorem is that on a certain subset $\Omega^* \subset \Omega$ which will be shown to be of $\mathbb{P}_\varepsilon$-measure one, the Hamiltonian $H_N^h(\varepsilon, \sigma)$ of the RDCW model can be approximated by the Hamiltonian $\tilde{H}_N^h(\sigma)$ of the standard CW model up to a small perturbation $\mathscr{H}_N^h(\varepsilon, \sigma) \equiv H_N^h(\varepsilon, \sigma) - \tilde{H}_N^h(\sigma)$ which uniformly in σ will be $o(N)$. Therefore, we will be allowed to give to the dilute model a standard mean-field treatment.

To determine the suitable set Ω^*, we proceed in the following way: let us consider the square array $\{\sigma_i \sigma_j\}_{(i,j) \in \Lambda \times \Lambda}$ whose elements are the products $\sigma_i \sigma_j$ of two spins for all possible pairs $(i, j) \in \Lambda \times \Lambda$. This aray is equivalently given by the partition of $\Lambda \times \Lambda$ into two subsets $\Lambda \times \Lambda = \Lambda_2^+(\sigma) \cup \Lambda_2^-(\sigma)$ containing respectively the "alligned" and "nonaligned" pairs of spins:

$$\begin{aligned} \Lambda_2^+(\sigma) &= \{(i, j) \in \Lambda \times \Lambda \mid \sigma_i \sigma_j = 1\} \\ \Lambda_2^-(\sigma) &= \{(i, j) \in \Lambda \times \Lambda \mid \sigma_i \sigma_j = -1\} \end{aligned} \tag{2.1}$$

Notice that the cardinality of the subsets $\Lambda_2^+(\sigma)$ and $\Lambda_2^-(\sigma)$ only depends on the variables $m_N(\sigma) = (1/N) \sum_{i=1}^N \sigma_i$:

$$\begin{aligned} |\Lambda_2^+(\sigma)| & \frac{1 + m_N^2(\sigma)}{2} N^2 \\ |\Lambda_2^-(\sigma)| &= \frac{1 - m_N^2(\sigma)}{2} N^2 \end{aligned} \tag{2.2}$$

Using this partition, we can rewrite the Hamiltonian (1.1) as

$$H_N^h(\varepsilon, \sigma) = -\frac{1}{2Np} \left\{ 2 \sum_{(i,j) \in \Lambda \times \Lambda} \varepsilon_{i,j} \chi_{\{(i,j) \in \Lambda_2^+(\sigma)\}} - \sum_{(i,j) \in \Lambda \times \Lambda} \varepsilon_{i,j} \right\} + h \sum_{i \in \Lambda} \sigma_i \tag{2.3}$$

where $\chi_{\{(i,j) \in \Lambda_2^+(\sigma)\}}$ is the characteristic function of the set $\Lambda_2^+(\sigma)$. Now let us define the subsets $\tilde{\Omega}_N \subset \Omega$ as the subsets for which the first two sums in (2.3) remains close to their mean value, i.e.,

$$\tilde{\Omega}_N \equiv \bigcap_{\sigma \in \Gamma^N} \left\{ \omega \in \Omega : \left| \sum_{(i,j) \in \Lambda \times \Lambda} \varepsilon_{i,j} \chi_{\{(i,j) \in \Lambda_2^+(\sigma)\}} - p |\Lambda_2^+(\sigma)| \right| \leqslant \gamma p |\Lambda_2^+(\sigma)| \right\} \tag{2.4}$$

where $\gamma \equiv \gamma(N)$ is a decreasing function of N such that $\gamma(N) \downarrow 0$ as $N \uparrow \infty$. Then if γ is appropriately chosen we have that for N large enough, and how large will depend on the sample, almost all ω will belong to $\tilde{\Omega}_N$. More precisely, defining the subset $\Omega^* \subset \Omega$ as

$$\Omega^* = \{ \omega \in \Omega : \exists N_0 \text{ s.t. } \forall N \geqslant N_0, \omega \in \tilde{\Omega}_N \} \tag{2.5}$$

we have the following result.

Proposition 1. Let $p \in (0, 1)$ be a nonincreasing function of N such that $pN \uparrow \infty$ as $N \uparrow \infty$. Let γ be a positive, strictly decreasing function of N such that $\gamma(N) \geqslant 3/(pN)^{1/2}$ and $\gamma(N) \downarrow 0$ as $N \uparrow \infty$. Then

$$\mathbb{P}_\varepsilon(\Omega^*) = 1 \tag{2.6}$$

where Ω^* is defined in (2.5).

Remark. To avoid useless discussions the case $p \equiv 1$ in Proposition 1 has been eliminated since it corresponds to the standard CW model for which Theorem 1 is already known.

In order to prove the proposition we need the following lemma: let $\tilde{\Omega}_N^c$ denote the complement of $\tilde{\Omega}_N$ in Ω; then we have the following result.

Lemma 2.1. Let p and γ be defined as in Proposition 1. Then, for N large enough,

$$\mathbb{P}_\varepsilon(\tilde{\Omega}_N^c) \leqslant c_0\sqrt{N}\,[\exp(-Nc^+) + \exp(-Nc^-)] \tag{2.7}$$

where c_0, c^+, and c^- are strictly positive constants.

Proof. By definition

$$\mathbb{P}_\varepsilon(\tilde{\Omega}_N^c) = \mathbb{P}_\varepsilon\left(\bigcup_{\sigma\in\Gamma^N}\left\{\omega\in\Omega\colon \left|\sum_{(i,j)\in\Lambda\times\Lambda}\varepsilon_{i,j}\chi_{\{(i,j)\in\Lambda_2^+(\sigma)\}} - p\,|\Lambda_2^+(\sigma)|\right| \geqslant \gamma p\,|\Lambda_2^+(\sigma)|\right\}\right) \tag{2.8}$$

which is bounded by

$$\mathbb{P}_\varepsilon(\tilde{\Omega}_N^c) \leqslant \sum_{\sigma\in\Gamma^N}\left\{\mathbb{P}_\varepsilon\left(\sum_{(i,j)\in\Lambda\times\Lambda}\varepsilon_{i,j}\chi_{\{(i,j)\in\Lambda_2^+(\sigma)\}} \geqslant p(1+\gamma)\,|\Lambda_2^+(\sigma)|\right) + \mathbb{P}_\varepsilon\left(\sum_{(i,j)\in\Lambda\times\Lambda}\varepsilon_{i,j}\chi_{\{(i,j)\in\Lambda_2^+(\sigma)\}} \leqslant p(1-\gamma)\,|\Lambda_2^+(\sigma)|\right)\right\} \tag{2.9}$$

Using now the exponential Markov inequality[3,22] and remembering that $\varepsilon_{i,j}$ are i.i.d., we get

$$\mathbb{P}_\varepsilon\left(\sum_{(i,j)\in\Lambda\times\Lambda}\varepsilon_{i,j}\chi_{\{(i,j)\in\Lambda_2^+(\sigma)\}} \geqslant p(1+\gamma)\,|\Lambda_2^+(\sigma)|\right) \leqslant \inf_{t\geqslant 0}\exp\{-|\Lambda_2^+(\sigma)|\,[\,p(1+\gamma)t - \ln \mathbb{E}_\varepsilon(e^{\varepsilon_{i,j}t})]\} \tag{2.10}$$

which by a direct calculation leads to

$$\mathbb{P}_\varepsilon\left(\sum_{(i,j)\in\Lambda\times\Lambda}\varepsilon_{i,j}\chi_{\{(i,j)\in\Lambda_2^+(\sigma)\}} \geqslant p(1+\gamma)\,|\Lambda_2^+(\sigma)|\right) \leqslant \exp[-|\Lambda_2^+(\sigma)|\,I_p^{(1)}(p(1+\gamma))] \tag{2.11}$$

where $I_p^{(1)}$ is defined on $[0, 1]$ by

$$I_p^{(1)}(x) = x\ln\left(\frac{x}{p}\right) + (1-x)\ln\left(\frac{1-x}{1-p}\right) \tag{2.12}$$

Similarly we have the bound

$$\mathbb{P}_\varepsilon\left(\sum_{(i,j)\in\Lambda\times\Lambda}\varepsilon_{i,j}\chi_{\{(i,j)\in\Lambda_2^+(\sigma)\}} \leqslant p(1-\gamma)\,|\Lambda_2^+(\sigma)|\right) \leqslant \exp[-|\Lambda_2^+(\sigma)|\,I_p^{(1)}(p(1-\gamma))] \tag{2.13}$$

Therefore, putting (2.12) and (2.13) together with (2.9) gives

$$\mathbb{P}_\varepsilon(\tilde{\Omega}_N^c) \leqslant \sum_{\sigma \in \Gamma^N} \{\exp[-|\Lambda_2^+(\sigma)|\, I_p^{(1)}(p(1+\gamma))] + \exp[-|\Lambda_2^+(\sigma)|\, I_p^{(1)}(p(1-\gamma))]\} \tag{2.14}$$

Now, making use of the fact that

$$|\Lambda_2^+(\sigma)| = \frac{1+m_N^2(\sigma)}{2} N^2$$

only depends on the variables $m_N(\sigma) = (1/N) \sum_{i=1}^N \sigma_i$, we can rewrite the sum

$$A = \sum_{\sigma \in \Gamma^N} \{\exp[-|\Lambda_2^+(\sigma)|\, I_p^{(1)}(p(1+\gamma))] + \exp[-|\Lambda_2^+(\sigma)|\, I_p^{(1)}(p(1-\gamma))]\} \tag{2.15}$$

as

$$A = \sum_{m \in \mathscr{S}_N} \binom{N}{\frac{1}{2}(1+m)N} \left\{ \exp\left[-\left(\frac{1+m^2}{2}\right) N^2 I_p^{(1)}(p(1+\gamma)) \right] + \exp\left[-\left(\frac{1+m^2}{2}\right) N^2 I_p^{(1)}(p(1-\gamma)) \right] \right\} \tag{2.16}$$

[Recall that $\mathscr{S}_N$ denotes the set of values the variable $m_N(\sigma)$ may take.] By the Stirling formula the binomial factor is equal to

$$\binom{N}{\frac{1}{2}(1+m)N} = \exp\left[-NI^{(2)}(m) + N\ln 2 - \frac{\ln N}{2} + r_N \right] \tag{2.17}$$

where $r_N = O(1/N)$ and $I^{(2)}$ is defined on $[-1, 1]$ by

$$I^{(2)}(x) = \frac{1-x}{2} \ln(1-x) + \frac{1+x}{2} \ln(1+x) \tag{2.18}$$

Therefore

$$\begin{aligned} A = {} & \exp\left\{ -N\left[\frac{N}{2} I_p^{(1)}(p(1+\gamma)) - \ln 2 \right] - \frac{\ln N}{2} + r_N \right\} \\ & \times \sum_{m \in \mathscr{S}_N} \exp\left[-\frac{N^2 m^2}{2} I_p^{(1)}(p(1+\gamma)) - NI^{(2)}(m) \right] \\ & + \exp\left\{ -N\left[\frac{N}{2} I_p^{(1)}(p(1-\gamma)) - \ln 2 \right] - \frac{\ln N}{2} + r_N \right\} \\ & \times \sum_{m \in \mathscr{S}_N} \exp\left[-\frac{N^2 m^2}{2} I_p^{(1)}(p(1-\gamma)) - NI^{(2)}(m) \right] \end{aligned} \tag{2.19}$$

To bound the sums in (2.19) let us notice that both $I_p^{(1)}$ and $I^{(2)}$ are positive convex functions. Moreover, $I^{(2)}(m)$ attains its infimum at $m=0$, so that

$$\sum_{m \in \mathscr{S}_N} \exp\left[-\frac{N^2 m^2}{2} I_p^{(1)}(p(1 \pm \gamma)) - N I^{(2)}(m)\right] \leqslant |\mathscr{S}_N| = N+1 \quad (2.20)$$

and

$$\mathbb{P}^{\varepsilon}(\tilde{\Omega}_N^c) \leqslant \frac{N+1}{\sqrt{N}} (\exp r_N) \left(\exp\left\{-N\left[\frac{N}{2} I_p^{(1)}(p(1+\gamma)) - \ln 2\right]\right\}\right.$$
$$\left. + \exp\left\{-N\left[\frac{N}{2} I_p^{(1)}(p(1+\gamma)) - \ln 2\right]\right\}\right) \quad (2.21)$$

In order to complete the proof, we are left to show that for p and γ defined as in Proposition 1 and for N large enough so that $p(1+\gamma) \in [p, 1]$ and $p(1-\gamma) \in [0, p]$, the exist two positive constants c^+ and c^- such that

$$\begin{aligned} &\tfrac{1}{2} N I_p^{(1)}(p(1+\gamma)) - \ln 2 \geqslant c^+ \\ &\tfrac{1}{2} N I_p^{(1)}(p(1-\gamma)) - \ln 2 \geqslant c^- \end{aligned} \quad (2.22)$$

To do so, let us rewrite $I_p^{(1)}(p(1+\gamma))$ and $I_p^{(1)}(p(1-\gamma))$ in the form

$$I_p^{(1)}(p(1+\gamma)) = p(1+\gamma)\ln(1+\gamma) + (1-p)\left(1 - \frac{p}{1-p}\gamma\right)\ln\left(1 - \frac{p}{1-p}\gamma\right) \quad (2.23)$$

and

$$I_p^{(1)}(p(1-\gamma)) = p(1-\gamma)\ln(1-\gamma) + (1-p)\left(1 + \frac{p}{1-p}\gamma\right)\ln\left(1 + \frac{p}{1-p}\gamma\right) \quad (2.24)$$

Now, using the series expansion of the logarithm, we have

$$\begin{aligned} &(1+x)\ln(1+x) = x + \sum_1^{\infty} \frac{(-1)^{n+1} x^{n+1}}{n(n+1)} \\ &(1+x)\ln(1-x) = -x + \sum_1^{\infty} \frac{x^{n+1}}{n(n+1)} \end{aligned} \qquad \text{for} \quad |x| < 1 \quad (2.25)$$

which implies the two following pairs of bounds:

$$\begin{aligned} &(1+x)\ln(1+x) \geqslant x + \frac{x^2}{2}\left(1 - \frac{x}{3}\right) \\ &(1-x)\ln(1-x) \geqslant -x \end{aligned} \quad (2.26)$$

and

$$(1+x)\ln(1+x) \geqslant x$$
$$(1-x)\ln(1-x) \geqslant -x+\frac{x^2}{2} \tag{2.27}$$

valid for any $x \in [0, 1]$. Since $p(1+\gamma) \in [p, 1]$ and $p(1-\gamma) \in [0, p]$, both γ and $[p/(1-p)]\gamma$ belong to $[0, 1]$. Thus, on one hand, (2.23) together with (2.26) gives

$$\begin{aligned} I_p^{(1)}(p(1+\gamma)) &\geqslant p\left[\gamma+\frac{\gamma^2}{2}\left(1-\frac{\gamma}{3}\right)\right]-(1-p)\left(\frac{p}{1-p}\gamma\right) \\ &= p\frac{\gamma^2}{2}\left(1-\frac{\gamma}{3}\right) \\ &\geqslant p\frac{\gamma^2}{3} \end{aligned} \tag{2.28}$$

while on the other hand, (2.24) and (2.27) give

$$I_p^{(1)}(p(1+\gamma)) \geqslant (1-p)\left(\frac{p}{1-p}\gamma\right)+p\left(-\gamma+\frac{\gamma^2}{2}\right)=p\ \frac{\gamma^2}{2} \tag{2.29}$$

Therefore we get the bounds

$$\begin{aligned} \frac{1}{2}NI_p^{(1)}(p(1+\gamma)) &\geqslant pN\frac{\gamma^2}{6} \\ \frac{1}{2}NI_p^{(1)}(p-\gamma)) &\geqslant pN\frac{\gamma^2}{4} \end{aligned} \tag{2.30}$$

and since γ decreases to zero more slowly than $3/(pN)^{1/2}$, there exist positive constants c^+ and c^- such that (2.22) holds. Thus the lemma is proven. ∎

Proof of Proposition 2. We want to show that $\mathbb{P}_\varepsilon(\Omega^*)=1$. By definition,

$$\mathbb{P}_\varepsilon(\Omega^*) \equiv 1-\mathbb{P}_\varepsilon((\Omega^*)^c) \tag{2.31}$$

and

$$(\Omega^*)^c=\{\omega \in \Omega: \forall N_0<\infty, \exists N \geqslant N_0 \text{ s.t. } \omega \in (\tilde{\Omega}_N)^c\} \tag{2.32}$$

and thus

$$0 \leqslant \mathbb{P}_\varepsilon((\Omega^*)^c)=\mathbb{P}_\varepsilon(\overline{\lim_{N\to\infty}}\ (\tilde{\Omega}_N)^c) \tag{2.33}$$

The Borel–Cantelli lemma[3] states that $\mathbb{P}_\varepsilon(\overline{\lim}_{N\to\infty}(\tilde{\Omega}_N)^c)=0$ if $\sum_N \mathbb{P}_\varepsilon((\tilde{\Omega}_N)^c)<\infty$ and by Lemma 2.1 this last condition holds and the proposition in proven. ∎

From now on we will consider that the function γ is chosen such that it satisfies the properties (i) and (ii) of Proposition 1. Returning now to the problem of bounding the Hamiltonian (1.1) and remembering that

$$\mathscr{H}_N^h(\sigma,\varepsilon)=H_N^h(\sigma,\varepsilon)-\tilde{H}_N^h(\sigma) \tag{2.34}$$

where $\tilde{H}_N^h$ is the Hamiltonian of the standard CW model, we have the following result.

Lemma 2.2. For all $\omega\in\Omega^*$ and all $\sigma\in\Gamma^N$

$$|\mathscr{H}_N^h(\sigma,\varepsilon)|\leqslant\tfrac{3}{2}\gamma N \tag{2.35}$$

Proof. This directly follows from the definition of $\tilde{\Omega}_N$ together with the decomposition (2.3) of the Hamiltonian $H_N^h(\sigma,\varepsilon)$. ∎

3. PROOF OF THE THEOREM

With the probabilistic display provided in the previous section, we are now ready to prove our main theorem. The essential idea is that on the set Ω^* the difference between the Hamiltonian and the averaged Hamiltonian is a small, uniformly in σ, that it does not contribute to the thermodynamic limit.

Proof of Part (i). By definition,

$$f_N^{\beta,h}(\varepsilon)=-\frac{1}{\beta N}\ln\sum_{\sigma\in\Gamma^N}\frac{1}{2^N}\exp\{-\beta[\tilde{H}_N^h(\sigma)+\mathscr{H}_N^h(\sigma)]\} \tag{3.1}$$

Now, for all $\omega\in\Omega^*$ Lemma 2.2 brings the bounds

$$e^{-3/2\beta N\gamma(N)}\leqslant e^{-\beta\mathscr{H}_N^h(\sigma)}\leqslant e^{3/2\beta N\gamma(N)} \tag{3.2}$$

Thus

$$|f_N^{\beta,h}(\varepsilon)-\tilde{f}_N^{\beta,h}|\leqslant\tfrac{3}{2}\gamma(N) \tag{3.3}$$

and the proof is completed by using Proposition 1. ∎

To prove parts (ii) and (iii) of Theorem 1 we need the following lemma:

Lemma 3.1. Let $\mathscr{Q}_N^{\beta,h}(\varepsilon)$ and $\tilde{\mathscr{Q}}_N^{\beta,h}$ be the measures on S_N induced, respectively, by $\mathscr{G}_N^{\beta,h}(\varepsilon)$ and $\tilde{\mathscr{G}}_N^{\beta,h}$ under the map

$$\begin{aligned} \Gamma^N &\to \mathscr{S}_N \\ \sigma &\mapsto m = m_N(\sigma) \end{aligned} \tag{3.4}$$

Then for all $\omega \in \Omega^*$ and all $m \in \mathscr{S}_N$

$$e^{-3/2\beta N\gamma(N)}\tilde{\mathscr{Q}}_N^{\beta,h}(m) \leqslant \mathscr{Q}_N^{\beta,h}(\varepsilon, m) \leqslant \tilde{\mathscr{Q}}_N^{\beta,h}(m)e^{+3/2\beta N\gamma(N)} \tag{3.5}$$

where

$$\tilde{\mathscr{Q}}_N^{\beta,h}(m) = \frac{\exp[-\beta N\tilde{F}^{\beta,h}(m) + r_N]}{\sum_{m\in\mathscr{S}_N}\exp[-\beta N\tilde{F}^{\beta,h}(m) + r_N]} \tag{3.6}$$

where $\tilde{F}^{\beta,h}(m)$ denotes the free energy functional of the standard CW model (see Appendix) and $r_N = O(1/N)$.

Proof of Lemma 3.1. By definition,

$$\begin{aligned} \mathscr{Q}_N^{\beta,h}(\varepsilon, m) &= \sum_{\substack{\sigma\in\Gamma^N:\\ m_N(\sigma)=m}} \mathscr{G}_N^{\beta,h}(\varepsilon, \sigma) \\ &= \sum_{\substack{\sigma\in\Gamma^N:\\ m_N(\sigma)=m}} \frac{\exp\{-\beta[\tilde{H}_N^h(\sigma) + \mathscr{H}_N^h(\varepsilon, \sigma)]\}}{\sum_{\sigma\in\Gamma^N}\exp\{-\beta[\tilde{H}_N^h(\sigma) + \mathscr{H}_N^h(\varepsilon, \sigma)]\}} \end{aligned} \tag{3.7}$$

and

$$\tilde{\mathscr{Q}}_N^{\beta,h}(m) = \sum_{\substack{\sigma\in\Gamma^N\\ m_N(\sigma)=m}} \frac{\exp[-\beta\tilde{H}_N^h(\sigma)]}{\sum_{\sigma\in\Gamma^N}\exp[-\beta\tilde{H}_N^h(\sigma)]} \tag{3.8}$$

Therefore, for all $\omega \in \Omega^*$, the bounds (3.5) are obtained by inserting (3.2) in (3.7) and using (3.8). Now, since $\tilde{H}_N^h(\sigma)$ only depends on the variables $m_N(\sigma)$, (3.8) can be written as

$$\tilde{\mathscr{Q}}_N^{\beta,h}(m) = \binom{N}{\frac{1}{2}(1+m)N} e^{-\beta N(m^2/2 + hm)} \Big/ \sum_{m\in\mathscr{S}_N} \binom{N}{\frac{1}{2}(1+m)N} e^{-\beta N(m^2/2 + hm)} \tag{3.9}$$

which together with (2.17) gives (3.6). ∎

Proof of Part (ii). It is enough to show that, for any continuous bounded function $g \in \mathscr{C}^b(\mathscr{S}_N, \mathbb{R})$ and all $\omega \in \Omega^*$

$$\lim_{h\downarrow 0}\lim_{N\uparrow\infty}\sum_{m\in\mathscr{S}_N} g(m)\,\mathscr{Q}_N^{\beta,h}(\varepsilon, m) = \begin{cases} g(0) & \text{if } 0\leqslant\beta\leqslant 1 \\ g(\tilde{m}^{\beta,h}) & \text{if } \beta\geqslant 1 \end{cases} \tag{3.10}$$

To do so, we denote by $\tilde{m}^{\beta,h}$ the unique minimum of $\tilde{F}^{\beta,h}$ and first introduce the set

$$B = \{m \in \mathscr{S}_N : |m - \tilde{m}^{\beta,h}| < \varrho(N)\} \tag{3.11}$$

Here the function $\varrho(N)$, which will be chosen appropriately later, is a decreasing function of N satisfying $\varrho(N) \downarrow 0$ as $N \uparrow \infty$. Next, we write

$$\begin{aligned} &\left| \sum_{m \in \mathscr{S}_N} [g(m) - g(\tilde{m}^{\beta,h})] \, \mathscr{Q}_N^{\beta,h}(\varepsilon, m) \right| \\ &\quad \leqslant \sum_{m \in B} |g(m) - g(\tilde{m}^{\beta,h})| \, \mathscr{Q}_N^{\beta,h}(\varepsilon, m) + \sum_{m \in B^c} |g(m) - g(\tilde{m}^{\beta,h})| \, \mathscr{Q}_N^{\beta,h}(\varepsilon, m) \end{aligned} \tag{3.12}$$

where B^c denotes the complement of B on $\mathscr{S}_N$. By continuity of g, for m in B and for any arbitrarily small ζ we have that $|g(m) - g(\tilde{m}^{\beta,h})| < \zeta$ provided that N is large enough. On the other hand, since g is bounded, $|g(m) - g(\tilde{m}^{\beta,h})| < 2 \|g\|_\infty$, where $\|g\|_\infty \equiv \sup_{m \in \mathscr{S}_N} |g(m)|$. Therefore

$$\left| \sum_{m \in \mathscr{S}_N} [g(m) - g(\tilde{m}^{\beta,h})] \, \mathscr{Q}_N^{\beta,h}(\varepsilon, m) \right| \leqslant \zeta + 2 \|g\|_\infty \sum_{m \in B^c} \mathscr{Q}_N^{\beta,h}(\varepsilon, m) \tag{3.13}$$

and we are left to show that, for $\omega \in \Omega^*$, the measure of B^c with respect to $\mathscr{Q}_N^{\beta,h}(\varepsilon, m)$ decreases to zero as N tends to infinity. By Lemma 3.1 we have, for N large enough,

$$\sum_{m \in B^c} \mathscr{Q}_N^{\beta,h}(\varepsilon, m) \leqslant c' \exp[2\beta N \gamma(N)] \frac{\sum_{m \in \mathscr{S}_N} \chi_{\{m \in B^c\}} \exp[-\beta N \tilde{F}^{\beta,h}(m)]}{\sum_{m \in \mathscr{S}_N} \exp[-\beta N \tilde{F}^{\beta,h}(m)]} \tag{3.14}$$

where $c' > 0$ is a constant. Now, remembering that $\tilde{m}^{\beta,h}$ realizes the minimum of $\tilde{F}^{\beta,h}$,

$$\begin{aligned} &\sum_{m \in B^c} \mathscr{Q}_N^{\beta,h}(\varepsilon, m) \leqslant c' \exp[2\beta N \gamma(N)] \\ &\qquad \times \sum_{m \in \mathscr{S}_N} \chi_{\{m \in B^c\}} \exp\{-\beta N [\tilde{F}^{u,h}(m) - \tilde{F}^{\beta,h}(\tilde{m}^{\beta,h})]\} \end{aligned} \tag{3.15}$$

and since there exists a strictly positive constant c such that

$$\tilde{F}^{\beta,h}(m) - \tilde{F}^{\beta,h}(\tilde{m}^{\beta,h}) \geqslant c(m - \tilde{m}^{\beta,h})^2 \tag{3.16}$$

the sum in the right-hand side of (3.15) is bounded by

$$\sum_{m \in \mathscr{S}_N} \chi_{\{m \in B^c\}} \exp\{-\beta N[[\tilde{F}^{\beta,h}(m) - \tilde{F}^{\beta,h}(\tilde{m}^{\beta,h})]\} \leqslant 2N \exp[-c\beta N \varrho^2(N)] \tag{3.17}$$

Finally, putting (3.17) together with (3.15) gives

$$\sum_{m \in B^c} \mathscr{Q}_N^{\beta,h}(\varepsilon, m) \leqslant 2c' \exp\left(-\beta N \left\{c\varrho^2(N) - \left[2\gamma(N) + \frac{\ln N}{\beta N}\right]\right\}\right) \tag{3.18}$$

and this last bound converges to zero as N tends to infinity that $\varrho(N)$ is chosen such that

$$c\varrho^2(N) > 2\gamma(N) + \frac{\ln N}{\beta N}(1+\eta) \tag{3.19}$$

for some real $\eta > 0$, which can be done for any c. Thus (ii) is proven by combining this result with (3.12), Proposition 1, and the fact that $\lim_{h \downarrow 0} \tilde{m}^{\beta,h} = \tilde{m}^{\beta,(+)}$. ▮

Proof of Part (iii). The proof of part (iii) essentially follows that of part (ii). We will only give the outline of the case $\beta > 1$; the case $\beta \leqslant 1$ is obtained following a similar scheme. We want to show that, for any continuous bounded function $g \in \mathscr{C}^b(\mathscr{S}_N, \mathbb{R})$ and all $\omega \in \Omega^*$

$$\lim_{N \uparrow \infty} \sum_{m \in \mathscr{S}_N} g(m)\, \mathscr{Q}_N^{\beta,0}(\varepsilon, m) = \tfrac{1}{2} g(\tilde{m}^{\beta,(+)}) + \tfrac{1}{2} g(\tilde{m}^{\beta,(-)}) \tag{3.20}$$

To do so, let us define the set B^+ and B^- as

$$\begin{aligned} B^+ &= \{m \in \mathscr{S}_N : |m - \tilde{m}^{\beta,(+)}| < \varrho(N)\} \\ B^- &= \{m \in \mathscr{S}_N : |m - \tilde{m}^{\beta,(-)}| < \varrho(N)\} \end{aligned} \tag{3.21}$$

where $\varrho(N)$ is a decreasing function of N which tends to zero as N tends to infinity. Then, decomposing the sum in (3.20) as

$$\begin{aligned} &\sum_{m \in \mathscr{S}_N} g(m)\, \mathscr{Q}_N^{\beta,0}(\varepsilon, m) \\ &\quad = \sum_{m \in B^{(+)}} g(\tilde{m}^{\beta,(+)})\, \mathscr{Q}_N^{\beta,0}(\varepsilon, m) + \sum_{m \in B^{(-)}} g(\tilde{m}^{\beta,(-)})\, \mathscr{Q}_N^{\beta,0}(\varepsilon, m) \\ &\qquad + \sum_{m \in B^{(+)}} [g(m) - g(\tilde{m}^{\beta,(+)})]\, \mathscr{Q}_N^{\beta,0}(\varepsilon, m) \\ &\qquad + \sum_{m \in B^{(-)}} [g(m) - g(\tilde{m}^{\beta,(-)})]\, \mathscr{Q}_N^{\beta,0}(\varepsilon, m) \\ &\qquad + \sum_{m \in (B^{(+)} \cup B^{(-)})^c} g(m)\, \mathscr{Q}_N^{\beta,0}(\varepsilon, m) \end{aligned} \tag{3.22}$$

we get

$$\left| \sum_{m \in \mathscr{S}_N} \{ g(m) - [g(\tilde{m}^{\beta,(+)}) \chi_{\{m \in B^{(+)}\}} + g(\tilde{m}^{\beta,(-)}) \chi_{\{m \in B^{(-)}\}}] \} \mathscr{Q}_N^{\beta,0}(\varepsilon, m) \right|$$

$$\leqslant \sum_{m \in B^{(+)}} |g(m) - g(\tilde{m}^{\beta,(+)})| \, \mathscr{Q}_N^{\beta,0}(\varepsilon, m)$$

$$+ \sum_{m \in B^{(-)}} |g(m) - g(\tilde{m}^{\beta,(-)})| \, \mathscr{Q}_N^{\beta,0}(\varepsilon, m)$$

$$+ \sum_{m \in (B^{(+)} \cup B^{(-)})^c} |f(m)| \, \mathscr{Q}_N^{\beta,0}(\varepsilon, m)$$

$$\leqslant \zeta + \|g\|_\infty \sum_{m \in (B^{(+)} \cup B^{(-)})^c} \mathscr{Q}_N^{\beta,0}(\varepsilon, m) \tag{3.23}$$

where by continuity of g, ζ can be made arbitrarily small provided that N is large enough. Splitting again the sum of the last term in (3.23), we get

$$\sum_{m \in (B^{(+)} \cup B^{(-)})^c} |g(m)| \, \mathscr{Q}_N^{\beta,0}(\varepsilon, m)$$

$$\leqslant \|g\|_\infty \left\{ \sum_{\substack{m \in (B^{(+)})^c \\ m \geqslant 0}} \mathscr{Q}_N^{\beta,0}(\varepsilon, m) + \sum_{\substack{m \in (B^{(-)})^c \\ m \leqslant 0}} \mathscr{Q}_N^{\beta,0}(\varepsilon, m) \right\} \tag{3.24}$$

and we have already seen that choosing $\varrho(N)$ appropriately, each of these sums converges exponentially fast to zero since for each half-space $\{m \in \mathscr{S}_N\colon m \geqslant 0\}$ and $\{m \leqslant 0\}$, $\tilde{m}^{\beta,(+)}$ and $\tilde{m}^{\beta,(-)}$ realize, respectively, the global minimum of $\tilde{F}^{\beta,0}$. Therefore we have shown that for $\omega \in \Omega^*$ the right-hand side of (3.23) converges to zero as N tends to infinity. To deal with the left-hand side, just notice that by symmetry

$$\lim_{N \to \infty} \sum_{m \in \mathscr{S}_N} \chi_{\{m \in B^{(+)}\}} \mathscr{Q}_N^{\beta,0}(\varepsilon, m) = \lim_{N \to \infty} \sum_{m \in \mathscr{S}_N} \chi_{\{m \in B^{(-)}\}} \mathscr{Q}_N^{\beta,0}(\varepsilon, m) = \tfrac{1}{2} \tag{3.25}$$

Thus, for $\beta > 1$, (iii) is proven. ∎

4. GENERALIZATIONS

It is clear that the proof of Theorem 1 does not really require all the assumptions we made on the random variables ε_{ij}. In fact, one only needs to check whether their distributions satisfy conditions allowing one to prove Propostion 1, i.e., in particular whether the large-deviation estimates (2.13) and (2.14) hold.

Here we want to exhibit sufficient conditions in two simple and illustrative contexts:

(i) Gaussian random variables.

(ii) Random variables for which the Bernstein conditions are satisfied.

From a physical point of view it is interesting to notice that these conditions will, in general, not imply that the couplings are ferromagnetic.

We will consider the following setting: Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space such that for all $N \in \mathbb{N}$ there exists a family $\varepsilon_N = \{\varepsilon_{i,j}(N)\}_{i=1,\dots,N; j=1,\dots,N}$ of i.i.d.r.v.'s on Ω taking value in $\mathbb{R}$ with distribution ρ_N, which is allowed to depend on N. We denote by $\mathbb{P}_{\varepsilon_N}$ the product measure with identical marginals ρ_N: $\mathbb{P}_{\varepsilon_N} = \prod_{(i,j)\in \Lambda\times\Lambda} \rho_N$.

Let us introduce the following two quantities that characterize ρ_N:

(i) Let $p_\rho(N)$ be the expectation with respect to ρ_N, i.e.,

$$p_\rho(N) = \int_{\mathbb{R}} \varepsilon \rho_N(d\varepsilon) \tag{4.1}$$

We will always assume $p_\rho(N)$ to be strictly positive, nonincreasing function of N.

(ii) Let $c_{\rho_N}(t)$ denote the functions

$$c_{\rho_N}(t) = \begin{cases} \log \mathbb{E}_{\rho_N}(\exp(t\varepsilon_{i,j})) & \text{if } \mathbb{E}_{\rho_N}(\exp(t\varepsilon_{i,j})) \text{ exists} \\ +\infty & \text{otherwise} \end{cases} \tag{4.2}$$

We now define the Hamiltonian $H_N^h(\varepsilon, \sigma)$ as in (2.1) through the couplings $J_{i,j} \equiv \varepsilon_{i,j}/2p_\rho(N)N$:

$$H_N^h(\varepsilon, \sigma) = -\frac{1}{2p_\rho(N)N} \sum_{(i,j)\in\Lambda\times\Lambda} \varepsilon_{i,j}\sigma_i\sigma_j - h\sum_{i\in\Lambda}\sigma_i \tag{4.3}$$

One then wants to find conditions on the distributions ρ_N under which, for N large enough, the Hamiltonian (4.3) can be written as a small perturbation of the standard CW Hamiltonian $\tilde{H}_N^h(\sigma)$ with probability with respect to $\mathbb{P}_{\varepsilon_N}$ exponentially close to one in N. To do so we proceed as in Section 2, defining for $N \in \mathbb{N}$ the set $\tilde{\Omega}_N$ as in (2.4):

$$\tilde{\Omega}_N = \bigcap_{\sigma\in\Gamma^N} \left\{ \omega\in\Omega: \left| \sum_{(i,j)\in\Lambda\times\Lambda} \varepsilon_{i,j}\chi_{\{(i,j)\in\Lambda_2^{(+)}(\sigma)\}} - p_\rho(N)\,|\Lambda_2^{(+)}(\sigma)| \right| \right.$$
$$\left. \leqslant \gamma p_\rho(N)\,|\Lambda_2^{(+)}(\sigma)| \right\} \tag{4.4}$$

where $\gamma \equiv \gamma(N)$ is a strictly decreasing function of N satisfying $\gamma(N) \downarrow 0$ as $N \uparrow \infty$. We have the following result.

Proposition 2. If there exists a function γ satisfying the above conditions such that for N large enough ρ_N satisfies

$$\begin{aligned} NI_{\rho_N}(p_\rho(N)(1+\gamma)) &> \ln 2 \\ NI_{\rho_N}(p_\rho(N)(1-\gamma)) &> \ln 2 \end{aligned} \tag{4.5}$$

where $I_{\rho_N}(x)$ is the Legendre–Fenchel transform of $c_{\rho_N}(t)$,

$$I_{\rho_N}(x) = \sup_{t \in \mathbb{R}} \{tx - c_{\rho_N}(t)\}, \qquad x \in \mathbb{R} \tag{4.6}$$

then there exists a strictly positive constant κ such that

$$\mathbb{P}_{\varepsilon_N}(\tilde{\Omega}_N) \geqslant 1 - e^{-\kappa N} \tag{4.7}$$

Before giving the proof of the proposition, we detail explicitly the conditions (4.5) on some examples. Notice that they clearly will require the existence of the Laplace transform of ρ_N for a sufficiently large range of t.

Example 1. Gaussian Couplings

Lemma 4.1. Let ρ_N be the normal distribution $\mathcal{N}(p_\rho(N), \sigma(N))$. If $Np_\rho^2(N)/\sigma^2(N) \uparrow \infty$ as $N \uparrow \infty$, then there exist functions γ such that (4.5) holds. Setting $\omega(N) \equiv [Np_\rho^2(N)/\sigma^2(N)]^{1/2}$, we can choose any function γ decreasing to zero more slowly than $[2\ln(2)]^{1/2}/\omega(N)$.

Proof. A standard calculation shows that $c_{\rho_N}(t) = \frac{1}{2}\sigma^2(N)t^2 + p_\rho(N)t$. For any real x the sup in (4.6) is attained at $t = [x - p_\rho(N)/\sigma^2(N)$ and $I_{\rho_N}(x) = [x - p_\rho(N)]^2/2\sigma^2(N)$. Moreover, $I_{\rho_N}(p_\rho(N)(1+\gamma)) = I_{\rho_N}(p_\rho(N)(1-\gamma))$. Thus the conditions (4.5) reduce to $Np_\rho^2(N)\gamma^2(N)/\sigma^2(N) > 2\ln(2)$. ∎

Remark. In the particular case $\sigma(N) \equiv 1$ the Hamiltonian (4.3) reads

$$H_N^h(\varepsilon, \sigma) = -\frac{1}{2\omega(N)\sqrt{N}} \sum_{(i,j) \in \Lambda \times \Lambda} \varepsilon_{i,j}\sigma_i\sigma_j - h \sum_{i \in \Lambda} \sigma_i \tag{4.8}$$

where ω tends to infinity as slowly as desired and $p_\rho(N)$ is allowed to go to zero as fast as $\omega(N)/\sqrt{N}$. This situation may look at first glance to be very close to the Sherrington–Kirkpatrick model of spin glasses, where we would have $p_\rho(N) \equiv 0$ and $\omega(N) = \omega \equiv \text{const}$. Our results of course show that with the constraints we have on $p_\rho(N)$ and $\omega(N)$, the properties of the model are already totally different from that of a true spin glass.

Remark. The conditions on the distributions ρ_N can in this case easily be translated in terms of the eigenvalues of the random matrix ε_N as the existence of a large enough gap between the largest and the second largest eigenvalue.

Example 2. Bernstein's Condition

Lemma 4.2. Let ρ_N be such that the centered variables $\varepsilon_{i,j} - p_\rho(N)$ satisfy the Bernstein condition,[17] i.e.,

$$\mathbb{E}_\rho(|\varepsilon_{i,j} - p_\rho(N)|^k) \leqslant \frac{k!}{2} \sigma^2(N) c^{k-2} \tag{4.9}$$

for some constant $0 < c < \infty$ and all $k \geqslant 2$. Then, if $Np_\rho^2(N)/\sigma^2(N) \uparrow \infty$ as $N \uparrow \infty$, and if there exists $0 < \alpha < 1$ such that

$$\frac{1}{N} \frac{4c^2}{\alpha^2} \frac{2-\alpha}{1-\alpha} \ln 2 \leqslant \sigma^2(N) \tag{4.10}$$

then (4.5) holds with γ chosen such that

$$\left[\ln 2 \frac{2-\alpha}{1-\alpha} \frac{\sigma^2(N)}{Np_\rho^2(N)}\right]^{1/2} \leqslant \gamma(N) \leqslant \frac{\alpha}{2c} \frac{\sigma^2(N)}{p_\rho(N)} \tag{4.11}$$

Proof. Note that

$$xt - c_\rho(t) = [x - p_\rho(N)]\, t - \ln \mathbb{E}\, e^{[\varepsilon - p_\rho(n)]\, t} \tag{4.12}$$

Now, using the Bernstein condition,[3] one gets that

$$\ln \mathbb{E}\, e^{[\varepsilon - p_\rho(N)]\, t} \leqslant \sigma^2(N)\, t^2 \frac{1}{2(1 - |tc|)} \tag{4.13}$$

for all t s.t. $|tc| < 1$. [Note that we could, of course, also impose the bound (4.13) in the lemma rather than the Bernstein condition.] Hence

$$xt - c_\rho(t) \leqslant [x - p_\rho(N)]\, t - \sigma^2(N)\, t^2 \frac{1}{2(1 - |tc|)} \tag{4.14}$$

Now denoting by $t^* \equiv t^*(x) = [x - p_\rho(N)]/\sigma^2(N)$ the value of t that realizes the supremum of the function $[x - p_\rho(N)]\, t - \sigma^2(N)\, t^2/2$, we get that

$$I_\rho(x) \geqslant \frac{[x - p_\rho(N)]^2}{\sigma^2(N)} \left(1 - \frac{1}{2(1 - |t^* c|)}\right) \tag{4.15}$$

Now, as long as $|t^*c| \leqslant \alpha/2$ with $\alpha < 1$, we have

$$I_\rho(x) \geqslant \frac{[x - p_\rho(N)]^2}{\sigma^2(N)} \frac{1-\alpha}{2-\alpha} \geqslant 0 \tag{4.16}$$

Thus

$$I_\rho(p_\rho(N)(1 \pm \gamma)) \geqslant \frac{p_\rho^2(N)\gamma^2}{\sigma^2(N)} \frac{1-\alpha}{2-\alpha} \tag{4.17}$$

and

$$|t^*(p_\rho(N)(1 \pm \gamma))| = \frac{p_\rho(N)\gamma}{\sigma^2(N)} \tag{4.18}$$

The condition $|t^*c| \leqslant \alpha/2$ now simply becomes

$$\gamma \leqslant \frac{\alpha}{2c} \frac{\sigma^2(N)}{p_\rho(N)} \tag{4.19}$$

and the conditions (4.5) reduce to

$$\frac{Np_\rho^2(N)\gamma^2}{\sigma^2(N)} \frac{1-\alpha}{2-\alpha} > \ln 2 \tag{4.20}$$

Conditions (4.19) and (4.20) now yield the bounds (4.11). ▮

After these examples we now come of the proof of Proposition 2.

Proof of Proposition 2. This proof is essentially identical to the part of the proof of Lemma 2.1 that leads to the bound (2.21). We will restrict ourselves to showing that the bound (2.11) becomes

$$\mathbb{P}_{\varepsilon_N}\left(\sum_{(i,j) \in \Lambda \times \Lambda} \varepsilon_{i,j} \chi_{\{(i,j) \in \Lambda_2^{(+)}(\sigma)\}} \geqslant p_\rho(N)(1+\gamma)\,|\Lambda_2^{(+)}(\sigma)| \right)$$
$$\leqslant \exp[-|\Lambda_2^{(+)}(\sigma)|\, I_{\rho_N}(p_\rho(N)(1+\gamma))] \tag{4.21}$$

where I_{ρ_N} is defined in (4.6). A similar proof yields

$$\mathbb{P}_{\varepsilon_N}\left(\sum_{(i,j) \in \Lambda \times \Lambda} \varepsilon_{i,j} \chi_{\{(i,j) \in \Lambda_2^{(+)}(\sigma)\}} \leqslant p_\rho(N)(1-\gamma)\,|\Lambda_2^{(+)}(\sigma)| \right)$$
$$\leqslant \exp[-|\Lambda_2^{(+)}(\sigma)|\, I_{\rho_N}(p_\rho(N)(1-\gamma))] \tag{4.22}$$

which replaces (2.13). First, using the exponential Markov inequality, we have

$$\mathbb{P}_{\varepsilon_N}\left(\sum_{(i,j)\in\Lambda\times\Lambda}\varepsilon_{i,j}\chi_{\{(i,j)\in\Lambda_2^{(+)}(\sigma)\}}\geqslant p_\rho(N)(1+\gamma)\,|\Lambda_2^{(+)}(\sigma)|\right)$$
$$\leqslant \inf_{t\geqslant 0}\exp\{-|\Lambda_2^{(+)}(\sigma)|\,[p_\rho(N)(1+\gamma)t-c_{\rho_N}(t)]\} \tag{4.23}$$

where $c_{\rho_N}(t)$ is defined in (ii). Next we want to show that

$$\sup_{t\geqslant 0}\{p_\rho(N)(1+\gamma)t-c_{\rho_N}(t)\}=I_{\rho_N}(p_\rho(N)(1+\gamma)) \tag{4.24}$$

To do so we need a well-known property of the function I_{ρ_N} (for a proof, see, e.g., ref. 5), namely that for any $x\in\mathbb{R}$, $I_{\rho_N}\geqslant 0$ and $I_{\rho_N}=0$ if and only if $x=p_\rho(N)$.

Now by Jensen's inequality $c_{\rho_N}(t)\geqslant p_\rho(N)$ for all $t\in\mathbb{R}$. Thus for t strictly negative and γ nonzero

$$p_\rho(N)(1+\gamma)t-c_{\rho_N}(t)\leqslant\{p_\rho(N)(1+\gamma)-p_0(N)\}\,t=p_\rho(N)\,\gamma t<0 \tag{4.25}$$

Therefore we see from the positivity of I_{ρ_N} that the supremum in the formula for I_{ρ_N} cannot occur for $t<0$. Finally, putting (4.24) together with (4.23) gives (4.21). ∎

From here on it is clear that all the results from Sections 3 and 4 carry over under the assumptions of Proposition 2.

APPENDIX

We give here the definitions and results for the standard Curie–Weiss model that we use throughout the paper. We refer to ref. 5, §5, where a complete study of the model is presented. With the notation of Section 1 and given the Hamiltonian

$$\tilde{H}_N^{\beta,h}(\sigma)=-\frac{1}{2N}\sum_{(i,j)\in\Lambda\times\Lambda}\sigma_i\sigma_j+h\sum_{i\in\Lambda}\sigma_i \tag{A.1}$$

the Curie–Weiss model is defined by the probability measure $\tilde{\mathscr{G}}_N^{\beta,h}$ which assigns to each configuration $\sigma\in\Gamma^N$ the probability

$$\tilde{\mathscr{G}}_N^{\beta,h}(\sigma)=\frac{\exp[-\beta H_N^{\beta,h}(\sigma)]}{2^N\tilde{Z}^{\beta,h}} \tag{A.2}$$

where $\tilde{Z}_N^{\beta,h}$ is the normalization $\sum_{\sigma\in\Gamma^N}\{\exp[-\beta H_N^{\beta,h}(\sigma)]\}/2^N$.

The following results are found:

(i) If $\tilde{f}_N^{\beta,h}$ denotes the finite-volume free energy,

$$\tilde{f}_N^{\beta,h} = -\frac{1}{\beta B}\ln \tilde{Z}_N^{\beta,h} \tag{A.3}$$

then the infinite-volume free energy $\tilde{f}_\infty^{\beta,h}$ is equal to

$$\tilde{f}_\infty^{\beta,h} = \inf_{m\in\mathbb{R}} \tilde{F}^{\beta,h}(m) \tag{A.4}$$

where the free energy functional $\tilde{F}^{\beta,h}$ is defined by

$$\tilde{F}^{\beta,h}(m) = \frac{1}{2}m^2 + hm - \frac{1}{\beta}I(m) \tag{A.5}$$

and

$$I(m) = \begin{cases} \dfrac{1-m}{2}\ln(1-m) + \dfrac{1+m}{2}\ln(1+m) & \text{if} \quad |m| \leqslant 1 \\ \infty & \text{if} \quad |m| > 1 \end{cases} \tag{A.6}$$

(ii) The points m giving the infimum in (A.4) are solutions of the equation

$$m = \tanh[\beta(m+h)] \tag{A.7}$$

For $0 < \beta \leqslant 1$, this equation has a unique solution $\tilde{m}^{\beta,h}$ which is zero for $h = 0$. For $\beta > 1$ and $h \neq 0$ it has a unique solution $\tilde{m}^{\beta,h}$ with the same sign as h. For $\beta > 1$ and $h = 0$ it has three solutions, $\tilde{m}^{\beta,(+)} > \tilde{m}^{\beta,h} = 0 > \tilde{m}^{\beta,(-)}$. Of these three solutions only $\tilde{m}^{\beta,(+)}$ and $\tilde{m}^{\beta,(-)}$ realize the infimum in (A.4).

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